convex sets with empty interiors can, in effect, be handled by the above results. The convexity requirement on X^0 is of course a restriction which cannot be dispensed with easily. (See however [Halkin 66, Canon et al. 66].) If we replace the convexity requirement on X^0 by the requirement that X^0 be open, a stronger necessary optimality condition than the above one can be obtained. In effect this will be an extension of the Fritz John stationary-point necessary optimality theorem *7.3.2* to the case of nonlinear equalities. We shall give this result in the next section of this chapter.

3. Fritz John and Kuhn-Tucker stationary-point necessary optimality criteria: X^0 open

We derive in this section necessary optimality criteria of the Fritz John and Kuhn-Tucker types from the minimum principle of the previous section.

Fritz John stationary-point necessary optimality theorem [Mangasarian-Fromovitz 67a]

Let X^0 be an open set in R^n. Let θ be a numerical function on X^0 let g be an m-dimensional vector function on X^0, and let h be a k-dimensional vector function on X^0. Let $\bar{x}$ be a (global) solution of the minimization problem

$$\theta(\bar{x}) = \min_{x \in X} \theta(x), \quad \bar{x} \in X = \{x \mid x \in X^0, g(x) \leqq 0, h(x) = 0\}$$

or a local solution thereof, that is,

$$\theta(\bar{x}) = \min_{x \in X \cap B_\delta(\bar{x})} \theta(x) \qquad \bar{x} \in X \cap B_\delta(\bar{x})$$

where $B_\delta(\bar{x})$ is an open ball around $\bar{x}$ with radius δ. Let θ and g be differentiable at $\bar{x}$, and let h have continuous first partial derivatives at $\bar{x}$. Then there exist $\bar{r}_0 \in R$, $\bar{r} \in R^m$, $\bar{s} \in R^k$ such that the following conditions are satisfied

$$\bar{r}_0 \nabla\theta(\bar{x}) + \bar{r}\nabla g(\bar{x}) + \bar{s}\nabla h(\bar{x}) = 0$$
$$\bar{r}g(\bar{x}) = 0$$
$$(\bar{r}_0, \bar{r}) \geqq 0$$
$$(\bar{r}_0, \bar{r}, \bar{s}) \neq 0$$

PROOF Let $\bar{x}$ be a global or local solution of the minimization problem.

In either case (since X^0 is open) there exists an open ball $B_\rho(\bar{x})$ around $\bar{x}$ with radius ρ such that $B_\rho(\bar{x}) \subset B_\delta(\bar{x}) \subset X^0$, and

$$\theta(\bar{x}) = \min_{x \in X^*} \theta(x) \qquad \bar{x} \in X^* = \{x \mid x \in B_\rho(\bar{x}), g(x) \leqq 0, h(x) = 0\}$$

Since $B_\rho(\bar{x})$ is a convex set with a nonempty interior, we have by the minimum principle *11.2.3* that there exist $\bar{r}_0 \in R$, $\bar{r} \in R^m$, $\bar{s} \in R^k$ such that

$$[\bar{r}_0\nabla\theta(\bar{x}) + \bar{r}\nabla g(\bar{x}) + \bar{s}\nabla h(\bar{x})](x - \bar{x}) \geqq 0 \qquad \text{for all } x \in B_\rho(\bar{x})$$
$$\bar{r}g(\bar{x}) = 0$$
$$(\bar{r}_0, \bar{r}) \geqq 0$$
$$(\bar{r}_0, \bar{r}, \bar{s}) \neq 0$$

Since for some small positive ζ

$$\bar{x} - \zeta[\bar{r}_0\nabla\theta(\bar{x}) + \bar{r}\nabla g(\bar{x}) + \bar{s}\nabla h(\bar{x})] \in B_\rho(\bar{x})$$

we have from the first inequality above that

$$\bar{r}_0\nabla\theta(\bar{x}) + \bar{r}\nabla g(\bar{x}) + \bar{s}\nabla h(\bar{x}) = 0 \quad ▌$$

To derive Kuhn-Tucker conditions from the above, we need to impose constraint qualifications on the problem.

The Kuhn-Tucker constraint qualification (see *7.3.3*)

Let X^0 be an open set in R^n, let g and h be m-dimensional and k-dimensional vector functions on X^0, and let

$$X = \{x \mid x \in X^0, g(x) \leqq 0, h(x) = 0\}$$

g and h are said to satisfy the *Kuhn-Tucker constraint qualification at* $\bar{x} \in X$ if g and h are differentiable at $\bar{x}$ and

$$\left.\begin{array}{r} y \in R^n \\ \nabla g_I(\bar{x})y \leqq 0 \\ \nabla h(\bar{x})y = 0 \end{array}\right\rangle \Rightarrow \left\langle\begin{array}{l} \text{There exists an } n\text{-dimensional vector function} \\ e \text{ on the interval } [0,1] \text{ such that} \\ a.\ e(0) = \bar{x} \\ b.\ e(\tau) \in X \text{ for } 0 \leqq \tau \leqq 1 \\ c.\ e \text{ is differentiable at } \tau = 0 \text{ and } \dfrac{de(0)}{d\tau} = \lambda y \\ \qquad\qquad \text{for some } \lambda > 0 \end{array}\right.$$

where

$$I = \{i \mid g_i(\bar{x}) = 0\}$$

3 **The weak Arrow-Hurwicz-Uzawa constraint qualification** (see *10.2.3*)

Let X^0 be an open set in R^n, let g and h be m-dimensional and k-dimensional vector functions on X^0, and let

$$X = \{x \mid x \in X^0, g(x) \leqq 0, h(x) = 0\}$$

g and h are said to satisfy the *weak Arrow-Hurwicz-Uzawa constraint qualification at* $\bar{x} \in X$ if g and h are differentiable at $\bar{x}$, h is both pseudoconvex and pseudoconcave at $\bar{x}$, and

$$\left\langle \begin{array}{l} \nabla g_Q(\bar{x})z > 0 \\ \nabla g_P(\bar{x})z \geqq 0 \\ \nabla h(\bar{x})z = 0 \end{array} \right\rangle \text{ has a solution } z \in R^n$$

where

$$P = \{i \mid g_i(\bar{x}) = 0, \text{ and } g_i \text{ is pseudoconcave at } \bar{x}\}$$

$$Q = \{i \mid g_i(\bar{x}) = 0, \text{ and } g_i \text{ is not pseudoconcave at } \bar{x}\}$$

4 **The weak reverse convex constraint qualification** (see *10.2.4*)

Let X^0 be an open set in R^n, let g and h be m-dimensional and k-dimensional vector functions on X^0, and let

$$X = \{x \mid x \in X^0, g(x) \leqq 0, h(x) = 0\}$$

g and h are said to satisfy the *weak reverse constraint qualification at* $\bar{x} \in X$ if g and h are differentiable at $\bar{x}$, and if for each

$$i \in I = \{i \mid g_i(\bar{x}) = 0\}$$

either g_i is pseudoconcave at $\bar{x}$ or linear on R^n, and h is both pseudoconvex and pseudoconcave at $\bar{x}$.

5 **The modified Arrow-Hurwicz-Uzawa constraint qualification** [Mangasarian-Fromovitz 67a]

Let X^0 be an open set in R^n, let g and h be m-dimensional and k-dimensional vector functions on X^0, and let

$$X = \{x \mid x \in X^0, g(x) \leqq 0, h(x) = 0\}$$

g and h are said to satisfy the *modified Arrow-Hurwicz-Uzawa constraint qualification at* $\bar{x} \in X$ if g is differentiable at $\bar{x}$, h is continuously differentiable at $\bar{x}$, $\nabla h_i(\bar{x})$, $i = 1, \ldots, k$, are linearly independent, and

$$\left\langle \begin{array}{l} \nabla g_I(\bar{x})z > 0 \\ \nabla h(\bar{x})z = 0 \end{array} \right\rangle \text{ has a solution } z \in R^n$$

where

$$I = \{i \mid g_i(\bar{x}) = 0\}$$

6 **Kuhn-Tucker stationary-point necessary optimality theorem**

Let X^0 be an open set in R^n, and let θ, g, and h be respectively a numerical function, an m-dimensional vector function, and a k-dimensional vector function, all defined on X^0. Let $\bar{x}$ be a (global) solution of the minimization problem

$$\theta(\bar{x}) = \min_{x \in X} \theta(x) \qquad \bar{x} \in X = \{x \mid x \in X^0, g(x) \leqq 0, h(x) = 0\}$$

or a local solution thereof, that is,

$$\theta(\bar{x}) = \min_{x \in X \cap B_\delta(\bar{x})} \theta(x) \qquad \bar{x} \in X \cap B_\delta(\bar{x})$$

where $B_\delta(\bar{x})$ is some open ball around $\bar{x}$ with radius δ. Let θ, g, and h be differentiable at $\bar{x}$, and let g and h satisfy

(i) *the Kuhn-Tucker constraint qualification 2 at $\bar{x}$, or*
(ii) *the weak Arrow-Hurwicz-Uzawa constraint qualification 3 at $\bar{x}$, or*
(iii) *the weak reverse convex constraint qualification 4 at $\bar{x}$, or*
(iv) *the modified Arrow-Hurwicz-Uzawa constraint qualification 5 at $\bar{x}$.*

Then there exist $\bar{u} \in R^m$ and a $\bar{v} \in R^k$ such that

$$\nabla\theta(\bar{x}) + \bar{u}\nabla g(\bar{x}) + \bar{v}\nabla h(\bar{x}) = 0$$
$$g(\bar{x}) \leqq 0$$
$$h(\bar{x}) = 0$$
$$\bar{u}g(\bar{x}) = 0$$
$$\bar{u} \geqq 0$$

PROOF (i)–(ii)–(iii) These parts of the theorem follow from Theorem *10.2.7*, parts (i), (ii), and (iii), by replacing $h(x) = 0$ in the above theorem by $h(x) \leqq 0$ and $-h(x) \leqq 0$.

(iv) All we have to show here is that $\bar{r}_0 > 0$ in the Fritz John theorem *1* above, for then we have that $\bar{x}, \bar{u} = \bar{r}/\bar{r}_0$ and $\bar{v} = \bar{s}/\bar{r}_0$ satisfy the Kuhn-Tucker conditions above. We assume now that $\bar{r}_0 = 0$ and exhibit a contradiction.

If $I = \{i \mid g_i(\bar{x}) = 0\}$ is empty, that is, there are no active constraints, then $\bar{r} = 0$ (since $\bar{r}g(\bar{x}) = 0$, $\bar{r} \geqq 0$ and $g(\bar{x}) \leqq 0$). Hence by Theorem *1*

$$\bar{s}\nabla h(\bar{x}) = 0 \qquad \bar{s} \neq 0$$

which contradicts the assumption of *5* that $\nabla h_i(\bar{x})$, $i = 1, \ldots, k$, are linearly independent. (If there are no equality constraints $h(x) = 0$, then $(\bar{r}_0,\bar{r}) = 0$ contradicts the condition $(\bar{r}_0,\bar{r}) \neq 0$ of Theorem *1*, since there is no $\bar{s}$.)

If I is not empty, then by Theorem *1* we have that

$$\bar{r}_I \nabla g_I(\bar{x}) + \bar{s}\nabla h(\bar{x}) = 0$$

$$\bar{r}_I \geqq 0$$

$$(\bar{r}_I,\bar{s}) \neq 0$$

If $\bar{r}_I = 0$, then $\bar{s} \neq 0$, and we have a contradiction to the assumption of *5* that $\nabla h_i(\bar{x})$, $i = 1, \ldots, k$, are linearly independent (if there is no $\bar{s}$, then $\bar{r}_I = 0$ implies $(\bar{r}_0,\bar{r}) = 0$, which contradicts the condition $(\bar{r}_0,\bar{r}) \neq 0$ of Theorem *1*). If $\bar{r}_I \neq 0$, then $\bar{r}_I \geq 0$. But by *5* there exists a z such that

$$\nabla g_I(\bar{x})z > 0 \qquad \text{and} \qquad \nabla h(\bar{x})z = 0$$

Hence

$$\bar{r}_I \nabla g_I(\bar{x})z + \bar{s}\nabla h(\bar{x})z > 0$$

which contradicts the equality above that

$$\bar{r}_I \nabla g_I(\bar{x}) + \bar{s}\nabla h(\bar{x}) = 0 \quad \blacksquare$$